On the Convergence of the Discrete Electrical Network Approximations for the Solution to the continuous Dirichlet Problem on the Unit Rectangle

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Abstract

This paper will prove the convergence of the discrete electrical network approximation for the solution to the continuous dirichlet problem of the unit rectangle by using the results of paper by Tyler Johnson and the notes by Profesoor Gunther Uhlmann.

1 Introduction

The motivation to this problem was on proving that if we are given a "discrete" boundary condition that takes values of the continuous bounary condition on at least finitely many points, then does the solution to the discrete dirichlet problem converge to the continuous dirichlet problem. This problem was explored in Tyler Johnson's 2012 REU paper, and half of the problem was solved with the proof of the norm of the continuous solution bounding above the limit supremum of the solution of the discrete solution. However, bounding below the limit infimum of the solution of the discrete solution by the continuous solution remained unsolved. By use of Professor Gunther Uhlman's results from his notes, the bounding has been proved. A much less generalized version of his proof will be provided below.

2 Notation

u_n := The "linear piecewise" γ-harmonic solution to the discrete dirichlet problem (i.e. $u_n = \phi_n$ at the boundary as described in Johnson's paper.)

$\tilde{u}_n$:= The γ-harmonic minimizer of the gamma norm with $\tilde{u}_n = \phi_n$ at the boundary.

u_∞ := The continuous γ-harmonic solution to the dirichlet problem.

ϕ_n := The boundary data for the discrete case.

ϕ := The boundary data for the continuous case, assume in C^1.

$||a||_\gamma$:= The gamma norm of a which is $\sqrt{(\int_{\partial\Omega} v^2 + \int_\Omega \gamma\Sigma(\frac{\partial v}{\partial x_j})^2)}$. (Note: this is slightly different from the definition in Johnson's paper, nevertheless, it will yield the same result in our case of proving the lim inf inequality)

R := The restriction map from $\mathbf{H}^1(\Omega)$ to $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$

$L_\gamma := \frac{\partial}{\partial x_i}\gamma_{ij}\frac{\partial}{\partial x_j}$ as defined in Uhlamann's notes, and shown to be the same as the map from H to its dual space.

3 Set Up

By Tyler Johnson's paper, we already have

$$||u_\infty||_\gamma \geq \overline{\lim_{n\to\infty}} ||u_n||_\gamma \tag{1}$$

Thus, if we prove

$$||u_\infty||_\gamma \leq \underline{\lim_{n\to\infty}} ||u_n||_\gamma \tag{2}$$

then we have

$$||u_\infty|| \leq \underline{\lim_{n\to\infty}} ||u_n||_\gamma \leq \overline{\lim_{n\to\infty}} ||u_n||_\gamma \leq ||u_\infty|| \tag{3}$$

which implies the limit exists and that

$$||u_\infty|| = \lim_{n\to\infty} ||u_n||_\gamma \tag{4}$$

To prove this, we will want to use Theorem 1.4 from Uhlmann's paper, which states that "The mapping

$$F : \mathbf{H}^1(\Omega) \to \mathbf{H}^{-1}(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{5}$$

defined by

$$Fu := \begin{pmatrix} L_\gamma u \\ Ru \end{pmatrix}, \tag{6}$$

is an isomorphism. That is, for any $f_1 \in \mathbf{H}^{-1}(\Omega)$ and$f_2 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ there exists a unique $u \in \mathbf{H}^1(\Omega)$ such that

$$Fu = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{7}$$

This solution u satisfies the estimate

$$||u||_{\mathbf{H}^1(\Omega)} \leq C(||f_1||_{\mathbf{H}^{-1}} + ||f_2||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}). \tag{8}$$

A brief outline of the proof in Uhlmann's paper.

Let us first prove that the mapping is in fact an isomorphism.

By the Riesz representation theorem, we know that L_γ is an isomorphism from $\mathcal{H}_0^1(\Omega)$ (the set which is the closure of H^1 equipped with γ-norm) to $\mathbf{H}^{-1}(\Omega)$. We will assume that the restriction map is an isomorphism and that by use of Poincare's inequality, it has a right bounded inverse (as Uhlmann's notes shows) for this paper.

For injectiveness: Suppose F(u) = 0 then R(u) = 0. This implies that u $\in H_0^1$ and that L_γ(u) is isomorphic to F(u) on $\mathcal{H}_0^1$. If R(u) = 0 and L_γ(u) = 0 and $u \in \mathcal{H}_0^1$ which implies that u =0. □

Now, to show that it is surjective, take $f_1 \in \mathbf{H}^{-1}(\Omega)$ and $f_2 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Then, pick a $w \in \mathbf{H}^1(\Omega)$ such that $R(w) = f_2$. Taking $v \in \mathcal{H}_0^1(\Omega)$ such that $L_\gamma v = f_1 - L_\gamma w$, where $L_\gamma v \in \mathbf{H}^{-1}(\Omega)$ we have

$$L_\gamma(v + w) = L_\gamma v + L_\gamma u = f_1 - L_\gamma u + L_\gamma u = f_1 \tag{9}$$
$$R(v + w) = R(v) + R(w) = 0 + R(w) = R(w) = f_2 \tag{10}$$

Thus, we see that F is bijective.

Now,

$F^{-1}Fu = u = F^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, hence $||u||_{\mathbf{H}^1(\Omega)} = ||F^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}||_{\mathbf{H}^1(\Omega)}$, thus we have $||u||_{\mathbf{H}^1(\Omega)} \leq C(||f_1||_{\mathbf{H}^{-1}(\Omega)} + ||f_2||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)})$.

4 Proof

Now, to prove that $||u_\infty||_\gamma \leq \underline{\lim}_{n\to\infty} ||u_n||_\gamma$

By definition, it is clear that

$$\tilde{u}_n \leq u_n$$

Hence, proving $||||u_\infty||_\gamma - ||\tilde{u}_n||_\gamma|| \to 0$ will suffice. To do this, let us note a few things:

Let us assume that there exists c and $C > 1$ such that $0 < c \leq \gamma \leq C$.

For continuous v_n we have

$$\begin{aligned} ||v_n||_\gamma^2 &\leq \int_{\partial\Omega} v_n^2 + C\int_\Omega \Sigma(\frac{\partial v_n}{\partial x_j})^2 \\ &\leq C(\int_{\partial\Omega} v_n^2 + \int_\Omega \Sigma(\frac{\partial v_n}{\partial x_j})^2) \\ &\leq C||v_n||^2_{H^1(\Omega)}. \end{aligned}$$

The same logic applies when using $c < \gamma$ so we have $c_1||u||_{H^1} \leq ||u||_\gamma \leq C_2||u||_{H^1}$.

By the theorem mentioned above, we have $||v_n||_{\mathbf{H}^1(\Omega)} \leq C(||L_\gamma v_n||_{\mathbf{H}^{-1}(\Omega)} + ||Rv_n||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)})$,

where $Fv_n = \begin{pmatrix} L_\gamma v_n \\ Rv_n \end{pmatrix}$. Setting $v_n = u_\infty - \tilde{u}_n$, we have that by the definition of the restriction map, $Rv_n = \phi - \phi_n$ and that because $u_\infty, \tilde{u}_n$ are both γ-harmonic and since L_γ is linear, $L_\gamma v_n = 0$.

Thus, we have $||v_n||_{\mathbf{H}^1(\Omega)} \leq C||Rv_n||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} = C||\phi - \phi_n||_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)} \leq C_2||\phi - \phi_n||_{\mathbf{H}^1}$.

$||\phi - \phi_n||^2_{\mathbf{H}^1} = \int_{\partial\Omega}(\phi - \phi_n)^2 + \int_\Omega \Sigma(\frac{\partial(\phi-\phi_n)}{\partial x_j})^2$.

Since $\phi, \phi_n \in C^1$, we have $||\phi - \phi_n||_\infty \to 0$ as $n \to 0$. Thus, we need only consider $\int_\Omega \Sigma(\frac{\partial(\phi-\phi_n)}{\partial x_j})^2$. $\phi - \phi_n$ may not be differentiable at finitely many points, however, intuitively, we can consider the contribution by these points to be 0, so using this let us look at the partial derivatives of $\phi - \phi_n$.

Since ϕ is defined meaningfully on the boundary, and since the boundary can be considered as a line, we will just consider the proof that for a continuous function in defined on the real line, the piecewise linear approximation of that function will have the derivative converge to that of the original function.

Let $I_k :=$ the intervals which make up our interval of interest. $\sum I_k = I$

Then, we have to show that $\int_I(\phi' - \phi'_n)^2 ds \to 0$ as $n \to \infty$ By construction, $\phi'_n = c_k$, where for $c_k, (1 \leq i \leq n)$ is a constant, and by the mean value theorem, we know that $\phi'(y) = c_k$ for some $y \in I_k$. Now, since ϕ' is continuous, we know that for all ϵ there exists δ such that $|x-y| < \delta$ implies $|\phi'(x) - \phi'(y)| < \epsilon$. Let us pick some ϵ. Then, there exists some N such that if we divide the interval into N equal pieces, and let x_{k-1}, x_k be the endpoints of the interval, $c_k = \frac{\phi(x_k)-\phi(x_{k-1})}{x_k - x_{k-1}}$ and that for some $y \in (x_{k-1}, x_k), \phi'(y) = c_k$. Furthermore, because of our choice of large enough N, we have $x_{k-1}, x_k \in N_{\delta(\epsilon)}(y)$ where$\delta(\epsilon)$is chosen such that $|p - q| < \delta(\epsilon)$ implies $|\phi'(s_k) - c_k| < \epsilon, \forall s_k \in I_k$, for $1 < k < n$. Then,

$$\begin{aligned}
\int_I (\phi' - \phi_n')^2 ds &= \sum \int_{I_k} (\phi' - \phi_n')^2 ds \\
&= \sum \int_{I_k} (\phi' - c_k)^2 ds \\
&\leq \sum \int_{I_k} (\epsilon)^2 ds \\
&= \epsilon^2 \sum \int_{I_k} ds \\
&<= \epsilon^2
\end{aligned}$$

Thus, we see that $\int_I (\phi' - \phi_n')^2 ds \to 0$ as $n \to \infty$. Therefore, $||\phi - \phi_n||^2_{\mathbf{H}^1} \to 0$ as $n \to \infty$, implying that $||u_\infty - \tilde{u}_n||_{\mathbf{H}^1(\Omega)} \to 0$ as $n \to \infty$. This gives us our desired result of $||u_\infty||_\gamma \leq \underline{\lim}_{n\to\infty} ||u_n||_\gamma$, hence the result of $||u_\infty||_\gamma = \lim_{n\to\infty} ||u_n||_\gamma$ as well. $\square$

5 References

[1] Curtis, B., and James A. Morrow. Inverse Problems for Electrical Networks. Series on applied mathematics Vol. 13. World Scientific, 2000.

[2] Johnson, Tyler. "Discrete Electrical Network Approximations for the Solution to the Continuous Dirichlet Problem on the Unit Rectangle ". 2012.

[3] Uhlmann, Gunther. Notes (?)